

# A novel type of spiral wave with trapped ions

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Pattern formation in ultra-cold quantum systems has recently received a great deal of attention. In this work, we investigate a two-dimensional model system accounting for the dynamics of trapped ions. We find a novel spiral wave which is rigidly rotating but with a peculiar core region in which adjacent ions oscillate in anti-phase. The formation of this novel spiral wave is ascribed to the novel excitability reported by Lee and Cross. The breakup of the novel spiral wave is probed and, especially, one extraordinary scenario of the disappearance of spiral wave caused by spontaneous expansion of the anti-phased core is unveiled.

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Spiral waves are the most frequently encountered pattern formation in two-dimensional systems far away from equilibrium. They are usually thought to be responsible for the patterns in a wide range of systems (nonlinear optics [1], magnetic films [2], new chemical systems [3], subcellular [4] biology, and complex plasma [5]). The underlying dynamics supporting spiral waves are usually oscillatory and excitable. Recently, spiral waves are also generated in media whose local dynamics is complex periodic or even chaotic [6, 7]. Spiral waves can display a number of distinct behaviors, some of which are quite complex. The simplest transition in spiral waves is Hopf bifurcation [8], which turns a rigidly rotating spiral wave into a quasi-periodic meandering one. Being another type of common transition, the breakup of spiral waves derives from two different ways: the core breakup and the far field breakup. In the core breakup, the spiral wave first develops into turbulence near the spiral core whose mechanism is due to the Doppler effects [9, 10]. In the far field breakup, the spiral wave first becomes unstable far away from the spiral core and the instability underlying it is the absolute Eckhaus one [11, 12].

As one type of wave propagation, spiral waves are always explored in systems where the spatial variable is continuous. However, spiral waves are also presented in systems of coupled oscillators where spatial variables are discrete. In the context of coupled oscillators, the states between adjacent oscillators are not required to be continuous, which may give rise to some impressing phenomena on the dynamics of spiral waves. Kuramoto. *et. al.* studied a two-dimensional non-locally coupled oscillators and found the existence of spiral wave chimera [13, 14]. In a spiral wave chimera, the oscillators in the core region of spiral wave are desynchronized while those around the periphery of the core are in synchronization. Martens *et. al.* analyzed the spiral wave chimera [15]. Yang *et. al.* studied a two-dimensional locally coupled Rössler oscilla-

tors [16]. They found a sandwiched spiral wave in which any two adjacent oscillators are in anti-phase and they attributed the presence of the sandwiched spiral wave to the shortwave instability of the homogeneous oscillation of the model [17, 18].

Recently, pattern formation in ultra-cold quantum systems which is of the nature of coupled system has recently received a great deal of attention. Lee and Cross considered a chain of ions [19], where dissipation is provided by laser heating and cooling and nonlinearity comes from trap anharmonicity and beam shaping. When the nonlinearities and interaction are small perturbations relative to the harmonic motion of ions, they derived an amplitude equation for the ions

$$\frac{dA_n}{dt} = ib(A_{n-1} + A_{n+1} - 2A_n) - (1 + ic)|A_n|^2 A_n + A_n.$$

$b$  is the coupling between adjacent trapped ions and  $c$  denotes how an ion's amplitude affects its harmonic frequency. The amplitude equation is similar to the complex Ginzburg-Landau equation (CGLE). Different from the CGLE which includes both reactive and dissipative interactions, the above amplitude equation contains only reactive interaction since the adjacent ions interact through reactive Coulomb force. Lee and Cross considered the pattern formation in the system described by Eq.(1); they found that long-wavelength waves are expected when  $bc > 0$  yet very short-wavelength waves for  $bc < 0$ . In the case of no-flux boundary condition, the only allowed pattern formation for  $bc > 0$  is homogeneous oscillation of all ions while very short-wavelength waves may display a more complicated structure for  $bc < 0$ . The most surprising discovery by Lee and Cross is that, when the homogeneous oscillation of ions in the chain for  $bc > 0$  is perturbed by a localized pulse of anti-phase oscillation, the perturbation would probably travel across the system for a long time before it dies off. This sort of excitability in an oscillatory medium is put down to the reactive interaction presented in the system. The new type of excitation of anti-phase perturbation to the homogeneous oscillation in Eq.(1) can also be observed

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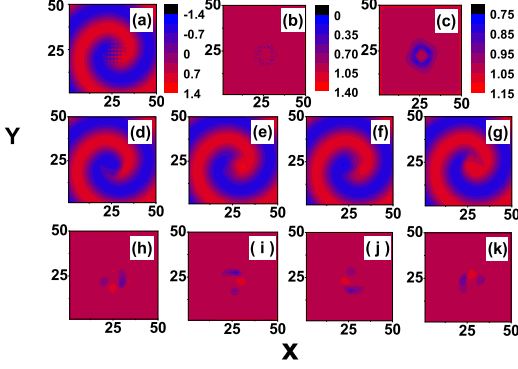


FIG. 1: (color online)  $b = 1$  and  $c = 0.2$ . (a) The wave pattern of SWAPC for  $Re(A_{i,j})$ . (b) The snapshot of  $|A_{i,j}|$ . (c) The snapshot of  $\langle |A_{i,j}| \rangle_t$ . The middle four panels show the snapshots of  $Re(A_{i,j})$  for different subsets and the bottom four show the snapshots of  $|A_{i,j}|$  for different subsets. (d) and (h) for the subset in which oscillators locate at positions with  $(2i, 2j)$ . (e) and (i) The subset with  $(2i, 2j + 1)$ . (f) and (j) The subset with  $(2i + 1, 2j)$ . (g) and (k) The subset with  $(2i + 1, 2j + 1)$ . The color scale used by (a) [or (b)] are for all plots of  $Re(A_{ij})$  (or  $|A_{ij}|$ ) in this work.  $N = 50$ .

when the system is generalized to a two-dimensional lattice.

Consider that, for a spiral wave in a two-dimensional coupled system, its phase singularity (or the rotation center) may be off lattice and adjacent oscillators on the opposite sides of the phase singularity are always in anti-phase, which indicates that there exists a persistent source for anti-phase perturbation. Then it will be of particular interest to delve how the new type of excitability reported by Lee and Cross influences the dynamics of spiral waves on a two-dimensional lattice. In this work, we will take this quest.

We use a square lattice where ions are trapped to nodes on it. The system is described as

$$\frac{dA_{i,j}}{dt} = ib(-4A_{i,j} + A_{i-1,j} + A_{i+1,j} + A_{i,j-1} + A_{i,j+1}) - (1 + ic)|A_{i,j}|^2 A_{i,j} + A_{i,j} \quad (1)$$

with  $i, j = 1, \dots, N$ . Open boundary conditions as in Ref. [19] are imposed upon the system.

To numerically simulate Eq.(1), we apply the fourth order Runge-Kutta method with  $dt = 0.001$ . A spiral wave occurs by initially setting the oscillators at the boundary with  $|A_{i,j}| = 1$  and the phases of  $A_{i,j}$  running from 0 to  $2\pi$  along the boundary.  $|A_{i,j}|$  for other oscillators are set to be 0. We first let  $b = 1$ ,  $c = 0.2$  and  $N = 50$ . Stunningly, a novel type of spiral wave, a rigidly rotating spiral wave with a core in which adjacent oscillators are in anti-phase, shows up.

For this novel type of spiral wave, two striking features can be revealed by the snapshots of the wave patterns of

the real part of  $A_{i,j}$  ( $Re(A_{i,j})$ ) in Fig. 1(a) and of the module of  $A(i, j)$  ( $|A_{i,j}|$ ) in Fig. 1(b). Firstly, the spiral wave possesses an odd core in which a pattern with the shortest wavelength appears [See Fig. 1(a)]; Secondly, there exists a transition annulus where the oscillation may be weaker (lower  $|A(i, j)|$ ). It is obvious that the transition annulus divides the wave pattern into the core region and the arm region [See Fig. 1(b)] and, in particular,  $|A_{i,j}|$  in the core region is as strong as that in the arm region. In Fig. 1(c), we exhibit the pattern for the time average of  $|A_{i,j}|$ . Together with Fig. 1(b), Fig. 1(c) illustrates that the novel type of spiral wave is stable, confirmed by the unchanged location of the core. To be mentioned, the dimension of the core of the spiral wave may be defined as that of the transition annulus. Furthermore, we pick up two pairs of adjacent oscillators: one pair locates in the core region and the other in the arm region. The time evolutions of these four oscillators are monitored. As shown in Figs. 2(a) and (b), all of them behave periodically. Nevertheless, it is interesting to find that the phase difference between adjacent oscillators within the core region is around  $\pi$ , which implies the anti-phase characteristic, and is also responsible for the pattern with the shortest wavelength in the core region. In contrast, there is only a minor phase shift between adjacent oscillators within the arm region, which is induced by the spiral wave propagation. In other words, oscillators in the arm region perform an in-phased oscillation. In addition, Figs. 2(a) and (b) show that the anti-phase oscillating is greatly faster than that of the homogeneous one, which is also seen in Ref.[19]. Combining the above observations together, we draw that the novel spiral wave in Fig. 1(a) is a rigidly rotating spiral wave with an anti-phased spiral core (we denote it as SWAPC).

In a CGLE, a rigidly rotating spiral wave always takes the following form:  $A(r, t) = F(r)\exp\{i[m\theta + \psi(r) - \omega t]\}$  and  $F(r) = 0$  at the rotation center (or the phase singularity) of the spiral wave [20]. In other systems such as reaction-diffusion systems, the similar formulation can be found provided that the spiral wave is rigidly rotating. Whereas, the spiral wave presented in Fig. 1(a) and (b) does not follow this formulation in the core region. Particularly, the rotation center of this spiral wave is replaced by an anti-phased spiral core and the phase singularity of a normal spiral wave is lost. The statement is supported by the patterns of both  $Re(A_{i,j})$  and  $|A_{i,j}|$  for four subsets of the system: the subset consisting of all oscillators with location  $(2i, 2j)$  ( $i, j = 1, 2, \dots, N/2$ ), the subset with  $(2i + 1, 2j)$ , the subset with  $(2i, 2j + 1)$ , and the subset with  $(2i + 1, 2j + 1)$ . As shown in Fig. 1(d)-(g), each subset displays a clear spiral wave pattern which shows that the anti-phased oscillation of  $Re(A_{i,j})$  in the core region is modulated by the in-phased oscillation in the arm region. Nevertheless,  $|A_{i,j}|$  in Fig. 1(h)-(k) for each subset substantiates that the phase singularity for a normal spiral wave is lost. Any subset itself possess

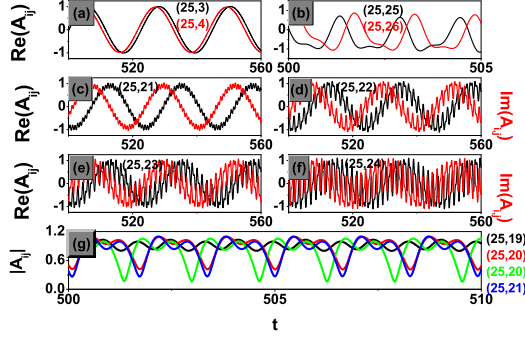


FIG. 2: (color online) The time evolutions of different oscillators locating at a line which crosses the core region of a SWAPC.  $b = 1$  and  $c = 0.2$ . The subscripts  $(i, j)$  in these plots denote the positions of the oscillators. (a) A pair of oscillators in the arm region. (b) A pair of oscillators in the core region. (c)-(f) The successive oscillators in the transition annulus. The black curves are for  $Re(A_{i,j})$  and the red ones for  $Im(A_{i,j})$ . (g) The evolutions of  $|A_{i,j}|$  for the oscillators in (c)-(f).

no phase singularity characterized by  $|A_{i,j}| = 0$  at the center of the spiral core in spite of the unique singularity a normal rigidly rotating spiral wave have. For each subset, two patches with low  $|A_{i,j}|$ , which situate oppositely to the center of the spiral core, turn up. It is no other than these patches in all subsets that contribute to the formation of the transition annulus.

To further figure out the phase singularity for SWAPC, we regard how the in-phased oscillation in the arm region passes through the transition annulus to the anti-phased oscillation in the core region along a line extending transversely. We present the time evolutions of successive oscillators on a line down the crossover area. The results given in Fig. 2(c)-(f) show distinctly two periodic components in each oscillator. When the oscillator is close to the arm region, it is dominated by the in-phased oscillation which is superimposed with a weak anti-phased oscillation. As the oscillator approaches the core region, the component of the anti-phased oscillation grows stronger and stronger and it becomes predominant. Recalling that the phase singularity of a normal spiral wave is actually manifested in the in-phased oscillation, it is the replacement of the in-phased oscillation by the anti-phased oscillation in the core region that causes the traditional phase singularity to be lost for SWAPC. To be mentioned, low  $|A_{i,j}|$  ( $|A_{i,j}| \simeq 0$ ) in the transition annulus is not related to the phase singularity of a spiral wave. As seen from Fig. 2(g), low  $|A_{i,j}|$  in the transition annulus just occurs when the anti-phased oscillation becomes comparable to the in-phased one. Especially, Fig. 2(g) shows that  $|A_{i,j}| \simeq 0$  in the transition annulus emerges with an equivalent period as that of the anti-phased oscillation, which states that the existence of low  $|A_{i,j}|$  in the tran-

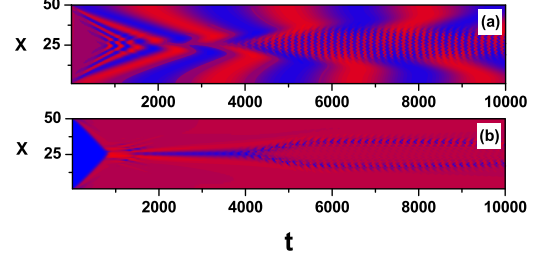


FIG. 3: (color online) The spatial-temporal plots of  $Re(A_{i,j})$  in (a) and  $|A_{i,j}|$  in (b) for the development of SWAPC.  $b = 1$  and  $c = 0.2$ .

sition annulus roots in the anti-phased oscillation and is foreign to the phase singularity. In short, the phase singularity in a normal spiral wave vanishes for SWAPC and, instead, an anti-phased core region plays the role of a rotor to support the spiral wave propagation.

Emphatically, though the wave pattern in Fig. 1(a) looks like the spiral wave chimera, they are essentially different. Firstly, the phases of adjacent oscillators in the core region in a spiral wave chimera state are unrelated while the phase difference between adjacent oscillators in SWAPC is kept around  $\pi$ . Secondly, as discussed in Ref.[15], the form  $A(r, t) = F(r)exp\{i[m\theta + \psi(r) - \omega t]\}$  is recovered for a spiral wave chimera provided that  $A(r, t)$  is replaced by an order parameter  $R(r, t)$ . That is, there exists a well defined phase singularity in the spiral wave chimera. However, it is quite different for SWAPC since the ordinary phase singularity has been substituted by an anti-phased core region where oscillation amplitude is as strong as that in the arm region.

To get more insight into how the excitability of anti-phased perturbation to a homogeneous oscillation leads to the formation of SWAPC, we monitor the evolution of  $Re(A_{i,j})$  and  $|A_{i,j}|$  of a chain of oscillators crossing the spiral wave core from the very initial stage when the novel spiral wave begins to build. Both spatial-temporal plots of  $Re(A_{i,j})$  and  $|A_{i,j}|$  in Fig. 3 show a general scenario towards SWAPC. An ordinary spiral wave is generated originally; Then, near the phase singularity of the normal spiral wave, an anti-phased perturbation comes into being; Ultimately, an anti-phased region with unchanged size appears and the novel spiral wave is established. The elaboration above explicitly demonstrates that the excitability of the anti-phased perturbation from the in-phased oscillation plays crucial roles on the formation of SWAPC.

SWAPC could exist within a large parameter range when  $b$  and  $c$  alter. The variation of  $b$  or  $c$  affects on the dynamics of SWAPC distinctly. Augmented  $b$  always leads to the expansion of the anti-phased core and longer wavelength in the arm region. Oppositely, with fixed

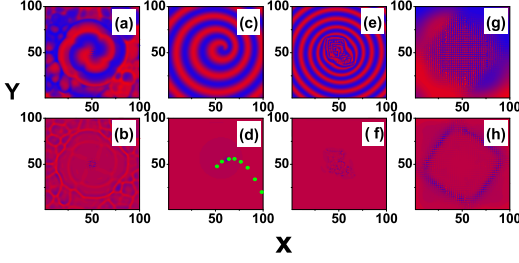


FIG. 4: (color online) Different types of instability of SWAPC. The top panels show the snapshots of  $Re(A_{i,j})$  and the bottom  $|A_{i,j}|$ . (a) and (b) are the far-away breakup of SWAPC owing to Eckhaus instability.  $b = 1$  and  $c = -0.33$ . (c) and (d) are the evanesce of SWAPC resulting from spiral wave drifting. The green dots in (d) designates the trajectory of the successive phase singularities.  $b = 1$  and  $c = 0.8$ . (e) and (f) are the novel near-core breakup of SWAPC due to the spontaneous expansion of the anti-phased core.  $b = 0.25$  and  $c = 0.2$ . (g) and (h) are the disappearance of SWAPC on account of the larger anti-phased core.  $b = 5$  and  $c = 0.2$ . The size of the system  $N = 100$ .

$b$ , both the size of anti-phased core and wavelength in the arm region are independent of the variation of  $c$ . In addition, either an outwardly propagating SWAPC or an inwardly propagating SWAPC wave [21] can be observed depending on  $b$  and  $c$ .

To further investigate the dynamics of SWAPC, we investigate how it breaks up which is always an interesting issue in the field of pattern formation. We consider a large system with  $N = 100$ . The results are insensitive to the system size (We have verified this prediction for different system sizes). In a normal CGLE, the spiral wave always breaks up through Eckhaus instability. However, depending on how the parameters vary, SWAPC in the system Eq. 1 may display rich scenes. In this work, merely several simple situations are taken into account. We prescribe  $b = 1$  at first. When  $c$  decreases beyond a critical value, the Eckhaus instability steps in and the normal far-away breakup of spiral wave appears. A snapshot is shown in Fig. 4(a) where a spiral wavelet with an anti-phased core is surrounded by a chaotic sea. The snapshot of the corresponding  $|A(i, j)|$  presented in Fig. 4(b) shows that there are many defects outside the spiral wavelet. As time goes on, the spiral wavelet will shrink further and it will die off when the anti-phased core is swallowed by the defect sea. To be mentioned, the final state after the disappearance of the spiral wavelet with the anti-phased core is portrayed by the anti-phase pattern. On the other hand, though increasing  $c$  may also cause SWAPC to vanish, it does not induce spiral wave breakup. Actually, when  $c$  is above a critical value, SWAPC is replaced by a normal spiral wave in the beginning; furthermore, to suppress the growth of

the anti-phased perturbation brought about by the phase singularity, the normal spiral wave has to drift spontaneously. Fig. 4(c) shows a snapshot of a drifting spiral wave with normal phase singularity. The trajectory of the drifting spiral wave is presented in Fig. 4(d) where the phase singularities at successive times are denoted on the plot. Clearly, the drifting spiral wave dies away in the end due to the collision between its phase singularity and the boundary. Then, we fix  $c$  while change  $b$ . For small  $b$ , one novel instability of SWAPC arises where the anti-phased core dilates spontaneously and persistently till overspreads the system and the arm region is eventually drive away. The snapshots of  $Re(A_{i,j})$  and  $|A_{i,j}|$  in Fig. 4(e) and (f) exhibit a SWAPC with a bulk of anti-phased core. Different from aforementioned circumstances, there is no instability of SWAPC is observed by increasing  $b$ . Nevertheless, there exist no SWAPC for sufficiently large  $b$  for any finite-size system since the ever swelling anti-phased core.

In summary, we find a novel type of spiral wave when studying the dynamics of trapped ions on the model system Eq.1: a rigidly rotating spiral wave with an anti-phased core (SWAPC). The formation of SWAPC is mainly due to the excitability discovered by Lee and Cross. Despite the dynamics of SWAPC as well as its breakup scenarios has been thoroughly investigated in this work, there still remain some open problems such as how does the size of the anti-phased core change and what comprehensive description is contained in the breakup picture. Furthermore, whether SWAPC exists in other cold atom quantum systems is also a fascinating topic.

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